Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(00)00080-0

J. Appl. Maths Mechs, Vol. 64, No. 4, pp. 547–557, 2000

0021-8928/00/$—see front matter

THE TWO-DIMENSIONAL MOTIONS OF A GAS WITH A SPECIAL ADIABATIC EXPONENT†

S. V. GOLOVIN

Novosibirsk

e-mail: sergey@hydro.nsc.ru

(*Received* 7 *October* 1999)

An invariant submodel of the two-dimensional equations of gas dynamics, constructed on an operator which is a combination of the time-shift, rotation and projective operators, is investigated using the PODMODELI program [1]. A canonical form of the submodel is constructed and a preliminary analysis of it is carried out (the group property, the hyperbolicity region and the first integrals). The self-similar solution of the submodel is investigated in detail. It determines the solutions of the submodel in question with closed invariant streamlines. Using a hierarchy of submodels, first integrals are obtained in the "second-level" submodel. A qualitative description of the nature of the motion is given (the contact characteristics and the particle trajectories). It is shown that the solution possesses discrete symmetry – invariance under rotation around the origin of coordinates by an angle that is a multiple of $2\pi/N$, with a certain natural N. It is pointed out that for certain values of the parameters, solutions of this type describe the gas motion with vacuum regions. The features of the flows obtained are illustrated by examples – the exact solutions of the gas – dynamic equations, which describe the expansion of a gas to a vacuum in an infinite time.

Another submodel, to construct which a projective symmetry is also used, was described in detail previously in [2].

1. PRELIMINARY DATA

In a polar system of coordinates, we will consider the two-dimensional gas-dynamic equations for a polytropic gas with adiabatic exponent $\gamma = 2$

$$\begin{gathered} D\mathbf{u} + \rho^{-1}\nabla p = r^{-1}(v^2, -uv), \quad D\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \quad DS = 0 \\ p = S\rho^2, \quad D = \partial_t + u\partial_r + r^{-1}v\partial_\theta \\ \nabla = (\partial_r, r^{-1}\partial_\theta), \quad \mathrm{div}\,\mathbf{u} = u_r + r^{-1}u + r^{-1}v_\theta \end{gathered} \tag{1.1}$$

Here u is the radial component and v is the tangential component of the velocity vector u, ρ is the density, p is the pressure and S is the entropy. These functions depend on the polar coordinates (r, θ) and the time t.

It is well known [3, 4], that Eqs (1.1) allow of a 10-dimensional Lie algebra L_{10}, which is an extension of the Galilean algebra by two extensions and a projective operator. It should be noted that the projective operator allows of gas-dynamic equations only for a chosen adiabatic exponent. Following Nikol'skii [5], we will call this adiabatic exponent special ($\gamma = 2$ for two-dimensional motions and $\gamma = 3$ for one-dimensional motions).

We will investigate the invariant submodel of Eqs (1.1), constructed by means of the operator

$$X = (t^2 + 1)\partial_t + tr\partial_r + a\partial_\theta + (r - tu)\partial_u - tv\partial_v - 4tp\partial_p - 2t\rho\partial_\rho$$

($a \geqslant 0$ is an arbitrary real parameter; for different values of a the operator X generates subalgebras that are non-conjugate in L_{10}). In accordance with the algorithm described earlier‡ the representation of the invariant solution can be written in the form

†*Prikl. Mat. Mekh.* Vol. 64, No. 4, pp. 569–579, 2000.

‡Ovsyannikov, L. V., Canonical form of invariant submodels of gas dynamics. Preprint No. 3-97, Inst. Gidrodinamiki, Sil. Otd., Ross. Akad. Nauk, Novosibirsk, 1997.

$$u = \frac{\bar{u}(\lambda,\varphi)+t\lambda}{\sqrt{t^2+1}}, \quad v = \frac{(\bar{v}(\lambda,\varphi)+a)\lambda}{\sqrt{t^2+1}}, \quad p = \frac{\bar{p}(\lambda,\varphi)}{(t^2+1)^2}, \quad \rho = \frac{\bar{\rho}(\lambda,\varphi)}{t^2+1}$$
$$S = \bar{S}(\lambda,\varphi), \quad \lambda = r/\sqrt{t^2+1}, \quad \varphi = \theta - a\,\mathrm{arctg}\,t \tag{1.2}$$

Substituting the representation of solution (1.2) into system (1.1) we obtain the canonical form of the submodel

$$\bar{D}\bar{u} + \bar{p}_\lambda/\bar{\rho} = ((a+\bar{v})^2-1)\lambda, \quad \bar{D}\bar{v} + \bar{p}_\varphi/(\bar{\rho}\lambda^2) = -2(a+\bar{v})\bar{u}/\lambda$$
$$\bar{D}\bar{\rho} + \bar{\rho}(\bar{u}_\lambda + \bar{v}_\varphi) = -\bar{\rho}\bar{u}/\lambda, \quad \bar{D}\bar{S} = 0, \quad \bar{p} = \bar{S}\bar{\rho}^2, \quad \bar{D} = \bar{u}\partial_\lambda + \bar{v}\partial_\varphi \tag{1.3}$$

2. ANALYSIS OF THE SUBMODEL. FIRST INTEGRALS

The third equation of (1.3) enables us to introduce an invariant stream function $\psi(\lambda,\varphi)$ by the relations

$$\psi_\lambda = -\lambda\bar{v}\bar{\rho}, \quad \psi_\varphi = \lambda\bar{u}\bar{\rho} \tag{2.1}$$

From the equations for the velocity components we then obtain, by a standard procedure, the invariant Bernoulli integrals and the entropies

$$\bar{u}^2 + \lambda^2\bar{v}^2 + (1-a^2)\lambda^2 + 4\bar{S}\bar{\rho} = F(\psi), \quad \bar{S} = S(\psi) \tag{2.2}$$

with arbitrary functions F and S. The vorticity ω for the gas-dynamic equations in a polar system of coordinates can be written in the form

$$\omega = -r^{-1}u_\theta + v_r + r^{-1}v$$

We will introduce the invariant vorticity

$$\bar{\omega} = \lambda\bar{v}_\lambda + 2(\bar{v}+a) - \lambda^{-1}\bar{u}_\varphi, \quad \omega = \bar{\omega}/(t^2+1)$$

The following invariant vorticity integral holds

$$\bar{\omega} - S'\bar{\rho}^2 = \bar{\rho}G(\psi) \quad (S' = dS/d\psi) \tag{2.3}$$

(G is an arbitrary function).

System (1.3) has a mixed elliptic-hyperbolic form. The characteristics will be sought in the form $h(\lambda,\varphi) = \text{const}$. It turns out that there is always a double characteristic $\bar{u}h_\lambda + \bar{v}h_\varphi = 0$ and two further characteristics are possible, defined by the equation

$$(\bar{u}h_\lambda + \bar{v}h_\varphi)^2 - \bar{c}^2(h_\lambda^2 + \lambda^{-2}h_\varphi^2) = 0 \tag{2.4}$$

(here $\bar{c}^2 = 2\bar{p}/\bar{\rho}$ is the square of the invariant velocity of sound). In order that the quadratic form in h_λ and h_φ on the left-side of (2.4) should not be sign-definite, it is necessary that the determinant of the matrix of this quadratic form should be negative. A check of this requirement leads to the following assertion.

Lemma 1. System (1.3) is hyperbolic in the region defined by the inequality

$$\bar{c}^2 < \bar{u}^2 + \bar{v}^2\lambda^2 \tag{2.5}$$

3. GROUP PROPERTY. THE SELF-SIMILAR SOLUTION

The group classification of submodel (1.3) with respect to the parameter a shows that for all values of the parameter the permissible Lie algebra is isomorphic to the factoralgebra $\mathrm{Nor}_{L_{10}}\{X\}/\{X\}$ (here the notation $\{X\}$ denotes a Lie algebra generated by the operator X). It is when $a \neq 1$ that submodel (1.1) allows of the operators

$$H_1 = \partial_\varphi, \quad H_2 = \lambda\partial_\lambda + \bar{u}\partial_{\bar{u}} - 2\bar{\rho}\partial_{\bar{\rho}}, \quad H_3 = \bar{p}\partial_{\bar{p}} + \bar{\rho}\partial_{\bar{\rho}}$$

and when $a = 1$ also

$$H'_4 = \cos\varphi\partial_\lambda - \lambda^{-1}\sin\varphi\partial_\varphi - \bar{v}\sin\varphi\partial_{\bar{u}} + (\lambda^{-2}\bar{u}\sin\varphi - \lambda^{-1}\bar{v}\cos\varphi)\partial_{\bar{v}}$$
$$H'_5 = \sin\varphi\partial_\lambda + \lambda^{-1}\cos\varphi\partial_\varphi + \bar{v}\cos\varphi\partial_{\bar{u}} - (\lambda^{-2}\bar{u}\cos\varphi + \lambda^{-1}\bar{v}\sin\varphi)\partial_{\bar{v}}$$

Further we will consider the self-similar submodel of submodel (1.3), generated by the extension operator

$$\bar{H} = \lambda\partial_\lambda + \bar{u}\partial_{\bar{u}} + k\bar{p}\partial_{\bar{p}} + (k-2)\bar{\rho}\partial_{\bar{\rho}}$$

(k is an arbitrary real parameter). Substituting the representation of the solution

$$\bar{u} = U(\varphi)\lambda, \quad \bar{v} = V(\varphi), \quad p = P(\varphi)\lambda^k$$
$$\bar{\rho} = R(\varphi)\lambda^{k-2}, \quad \bar{S} = s(\varphi)\lambda^{-k+4} \tag{3.1}$$

into Eqs (1.3) we obtain the submodel equations

$$VV' + P'/R = -2(V+a)U, \quad VU' = (V+a)^2 - 1 - U^2 - kP/R$$
$$(VR)' = -kRU, \quad Vs' = (k-4)Us \tag{3.2}$$

Remark 1. It can be verified that the "two-step" submodel (3.2) (the submodel for submodel (1.3)) can be obtained directly from the gas-dynamic equations as a "single-step", by considering the invariant solution of Eqs (1.1) with respect to the subalgebra generated by the operators X and H, where

$$H = r\partial_r + u\partial_u + v\partial_v + kp\partial_p + (k-2)\rho\partial_\rho$$

This fact is an illustration of the Lie–Ovsyannikov–Talyshev lemma [6] on the hierarchy of invariant submodels. It turns out that, in addition to classification, it is also of practical value, namely, it provides the possibility "inheriting" the first integrals of submodel (1.3).

Consider the equations for the stream function (2.1). Substituting the representation of solution (3.1) we obtain

$$\psi_\lambda = -\lambda^{k-1}V(\varphi)R(\varphi), \quad \psi_\varphi = \lambda^k U(\varphi)R(\varphi) \tag{3.3}$$

When $k \neq 0$, by virtue of the third equation of (3.2), we hence obtain, apart from unimportant constants

$$\psi = \lambda^k|VR| \tag{3.4}$$

When $k = 0$, integrating the first equation of (3.3) with respect to λ we obtain

$$\psi = -\ln|\lambda|V(\varphi)R(\varphi) + \psi_0(\varphi)$$

Substituting into the second equation of (3.3) we obtain

$$-\ln|\lambda|(VR)' + \psi'_0 = UR$$

But since, when $k = 0$, it follows from the equations of submodel (3.2) that $(VR)' = 0$, we have

$$\psi_0 = \int U(\varphi)R(\varphi)d\varphi$$

From the last equation of (3.2) we have $UR = -VRs'/(4s)$. Then, since $(VR)' = 0$, we have, apart from an unimportant additive constant,

$$\psi_0 = -\tfrac{1}{4} VR \ln s$$

and hence we can choose as the stream function

$$\psi = \ln(s\lambda^4) \tag{3.5}$$

Having the representation for the stream function we can determine the specific form of the arbitrary functions S and F, which occur in the first integrals of system (1.3). We obtain the entropy integral

$$s^k = S_0|VR|^{4-k}$$

and the Bernoulli integral (α is an arbitrary constant)

$$U^2 + V^2 + 4sR + (1 - a^2) = \begin{cases} \alpha\,|VR|^{2/k}, & k \neq 0 \\ \alpha\sqrt{s}, & k = 0 \end{cases}$$

It is convenient to introduce the square of the invariant velocity of sound $Z(\varphi)$

$$c^2 = Zr^2/(t^2 + 1)^2, \qquad Z(\varphi) = 2P/R$$

Here the equations of submodel (3.2) are

$$\begin{aligned} &(Z - V^2)V' = -(k/2 + 2)ZU + 2(V + a)UV \\ &VU' = (V + a)^2 - 1 - U^2 - kZ/2 \\ &(Z - V^2)Z' = (k - 4)Z^2U/(2V) + 2(V - a)UZ \\ &Vs' = (k - 4)Us \end{aligned} \tag{3.6}$$

The last equation can be detached and solved independently after U and V are determined from the first three equations.

Using the function $Z(\varphi)$ we can rewrite the integrals obtained in the form

$$s = S_0\,|VZ|^{1-k/4} \tag{3.7}$$

$$U^2 + V^2 + 2Z + (1 - a^2) = \alpha\sqrt{|VZ|} \tag{3.8}$$

It can be seen from the last integral that there is an important difference in the nature of the flow described when $a^2 \geqslant 1$ and $a^2 < 1$. When $a^2 < 1$ the surface (3.8) in (U, V, Z) space always lies in the half-space $Z > 0$, and is separated from the $Z = 0$ plane. Hence, over the whole region the function Z, and together with it the invariant density R, are positive. At the same time, when $a = 1$ the surface (3.8) passes through the point $(U, V, Z) = (0, 0, 0)$, and when $a > 1$ may intersect the $Z = 0$ plane, which denotes the occurrence of a vacuum region in the region of the gas flow.

4. CONTACT CHARACTERISTICS. DESCRIPTION OF THE MOTION

The equation $\psi = \text{const}$ defines invariant streamlines for submodel (1.3). The standard streamline is given by the equation

$$\lambda = |V(\varphi)Z(\varphi)|^{-1/4} \tag{4.1}$$

all the remaining ones being homothetic. Note that, in view of the definition of the variable φ, the functions Z and V must be periodic with half-period $T = \pi/N$ (N is an integer), otherwise the solution will not be continuous over the whole (r, θ) plane. Hence from (4.1) we quickly obtain that when $V \neq 0$ submodel (3.2) defines the solutions of (1.3) with closed invariant streamlines. Similar flows with closed streamlines were obtained earlier [7, 8] for the equations of gas dynamics and an ideal incompressible fluid. Obviously solution (3.2) itself and streamline (4.1) possess discrete symmetry – invariance under rotation in the (λ, φ) plane by an angle of $2\pi/N$ about the origin of coordinates. We will further show that the invariant streamlines (4.1) in "physical" space correspond to it.

The equations of the trajectory of a particle, starting at the initial instant $t = 0$ from the position (r_0, θ_0), is found as the solution of the Cauchy problem (the formulation of the representation of the solution is used)

$$\frac{dr}{dt} = \frac{(U+t)r}{t^2+1}, \quad \frac{d\varphi}{dt} = \frac{V}{t^2+1}, \quad r(0) = r_0, \quad \varphi(0) = \theta_0 \tag{4.2}$$

Remark 2. For the differentiable function $V(\varphi)$ we have: if $V(\varphi_*) = 0$ for a certain $\varphi = \varphi^*$, then along the whole world line, which passes through the point (t, r, φ_*), the equation $V(\varphi) = 0$ holds. Conversely, if for a certain $\varphi = \varphi_*$ we have $V(\varphi_*) \neq 0$, then $V(\varphi) \neq 0$ along the whole world line, drawn through the point (t, r, φ_*). Moreover, the $\varphi = \varphi_*$: $V(\varphi_*) = 0$ plane is a contact characteristic. The proof follows from the theorem of the uniqueness of the solution of the Cauchy problem for an ordinary differential equation.

The integral of the first equation of (4.2) is the equation ψ = const, which, by virtue of the equality

$$D\psi = 0, \quad D = \partial_t + u\partial_r + r^{-1}v\partial_\theta$$

is a contact characteristic on the solutions described by this submodel. Rewriting the equation ψ = const in the initial variables, we obtain

$$r = \frac{A\sqrt{t^2+1}}{|VZ|^{1/4}}, \quad A = r_0\,|V(\theta_0)Z(\theta_0)|^{1/4} \tag{4.3}$$

Hence, the family of surfaces which are contact characteristics of the gas-dynamic equations on the solutions of submodel (3.2) correspond to the invariant streamlines ψ = const in the physical space $R^3(t, r, \theta)$. The surfaces of this family are obtained from one standard surface with $A = 1$ by the extension transformation $r \to Ar$ with suitable parameter A.

In order to obtain its level line $t = t_0$, we must turn streamline (4.1) by an angle a arctg t_0 anticlockwise and extend it by a factor of $\sqrt{(t_0^2 + 1)}$ with respect to the origin of coordinates. In particular, the level line $t_0 = 0$ coincides exactly with streamline (4.1).

The second equation of (4.2) can be integrated in quadratures

$$t = \operatorname{tg}\tau(\varphi, \theta_0), \quad \tau(\varphi, \theta_0) = \int_{\theta_0}^{\varphi} \frac{d\xi}{V(\xi)} \tag{4.4}$$

By virtue of Remark 2 $\tau(\varphi, \theta_0)$ is a monotonic function of φ, and, of course, the function $\varphi(t, \theta_0)$, obtained by inverting equality (4.4), will also be monotonic and, as can be noted, is bounded with respect to t. Reverting, using (1.2), to the physical variables, we obtain the equation of surfaces – the contact characteristics of the gas-dynamic equations, of the common position with respect to the family (4.3)

$$\theta(t, \theta_0) = \varphi(t, \theta_0) + a\operatorname{arctg} t \tag{4.5}$$

The surfaces of this family are marked by a ray, the vertex of which moves uniformly along the t axis, while it itself turns in accordance with the expression $\theta = \theta(t, \theta_0)$, remaining parallel to the $t = 0$ plane. In particular, it therefore follows that if at some instant of time the particles are on one ray, emerging from the origin of coordinates, then for any t they will also be on one ray.

The world line of each particle is obtained as the intersection of the corresponding contact characteristics (4.3) and (4.5). And of course, the motion of each particle is made up of two components – the motion along the ray and rotation together with its ray. Formula (4.3) corresponds to the first part and formula (4.5) corresponds to the second.

By virtue of the lemma on the density [9, p. 157] in continuous motion a line on which $\rho = 0$ (a vacuum line) is a contact characteristic. Hence, if at the point (t_0, r_0, θ_0) the density is non-zero, then on all the world lines drawn from this point the density, the pressure and the velocity of sound cannot vanish. And hence, it follows from Remark 2 and formula (4.3) that the particles depart an infinite distance from the origin of coordinates after an infinite time.

The invariant vorticity $\Omega(\varphi)$ is introduced by the relations

$$\omega = \Omega/(f^2 + 1), \qquad \Omega = 2(V + a) - U'$$

Using the second equation of (3.6) and the Bernoulli integral (3.8) we obtain

$$\Omega = \frac{\alpha\sqrt{VZ} + (k/2 - 2)Z}{V} \tag{4.6}$$

Further, we obtain from (3.6)

$$\frac{dZ}{dV} = \frac{(k-4)Z^2 + 4(V-a)VZ}{4(V+a)V^2 - (k+4)VZ} \tag{4.7}$$

When $a = 0$ this equation can be integrated in the form

$$V^{k-4}Z^{3k+4} = \beta(-4V^2 + (3k+4)Z)^{2k}, \quad \beta = \text{const}$$

while in the opposite case by making the replacement $h = Z/(aV)$ it is reduced to a Bernoulli equation, the solution of which can be expressed in terms of the hypergeometric function ${}_2F_1(-1/2, 3/4 + 1/k, 1/2; kh/2)$. For the particular value of the parameter $k = 4$ the solution of Eq. (4.7) can be expressed in terms of elementary functions

$$4Z^2 + \beta V^2 - 4aVZ - 4\beta Z + 2a\beta V + a^2\beta = 0, \quad \beta = \text{const} \tag{4.8}$$

Remark 3. Other values of the parameters for which the solution of Eq. (4.7) can be expressed in terms of elementary functions are, of course, also possible. For example, for $k = -4$ its integral has the form

$$V^2 Z = \beta(2VZ + aZ + V(V-a)^2), \quad \beta = \text{const}$$

However, they will not be considered here.

5. PARTICULAR SOLUTIONS

We will illustrate the method for the further investigation of system (3.6) for a particular value of the parameter $k = 4$. This case corresponds to isentropic gas flows. Integral (4.8), for different values of the parameter β, gives a family of second-order curves in the (V, Z) phase plane: when $\beta > a^2$ we have a family of ellipses, when $\beta < a^2$ and $\beta \neq 0$, we have hyperbola, and when $\beta < a^2$ and $\beta = 0$ we have a pair of straight lines $Z = 0$ and $Z = aV$. Finally, when $\beta = a^2$ Eq. (4.8) degenerates and gives the straight line

$$Z = a(V + a)/2 \tag{5.1}$$

The centre of the hyperbola and the ellipses obtained when $\beta \neq a^2$, is situated at the point $(V_0, Z_0) = (0, \beta/2)$. Moreover, curve (4.8) can intersect the $Z = 0$ axis only at those points when it degenerates into a straight line. The pattern of the integral curves (4.8) is shown in Fig. 1 for different values of the ratio β/a^2.

When curve (4.8) degenerates into a straight line, an explicit construction of the solution of system (3.6) is possible.

The case $\beta = a^2$. from Eq. (4.8) we obtain relation (5.1) between the square of the invariant velocity of sound Z and the peripheral component of the velocity V. By definition $Z \geqslant 0$, i.e. $V \geqslant -a$ by virtue of the fact that $a \geqslant 0$. Substituting the expressions for z from (5.1) and for U from (3.8) into the first equation of (3.6) and taking into account the limitations on the parameters imposed earlier, we obtain and equation for V

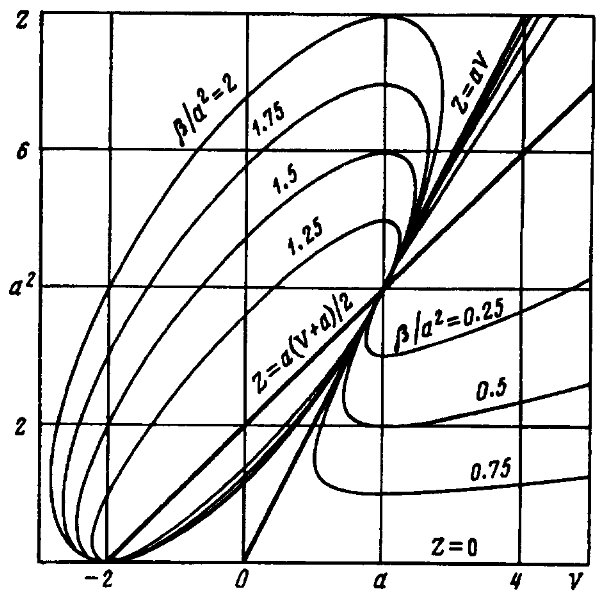

Fig. 1.

$$(V')^2 = \frac{2(V+a)^2 f(V)}{(V+a/2)^2}, \quad f(V) = \alpha\sqrt{|V(V+a)|} - V^2 - aV - 1 \tag{5.2}$$

In order for a periodic solution of Eq. (5.2) to exist it is necessary to choose the parameters of the problem so that intevals $[V_l, V_r]$ exist, at the ends of which the function $f(V)$ has simple roots, while inside it is positive. It is necessary to satisfy the requirement $Z \geqslant 0$, i.e. $V_1 \geqslant -a$. Moreover, the section (V_l, V_r) must not contain the point $V = -a/2$, since within this section the derivative $V'(\varphi)$ becomes infinite.

Lemma 2. The required section $[V_l, V_r]$ is constructed by the following algorithm: the left limit V_l is chosen arbitrarily from the interval $(0, V_+)$ $(V_\pm = (-a \pm \sqrt{(a^2+4)})/2$. Here the constant α is found from the equation $f(V_l) = 0$. The right limit V_r is found as the unique root of the equation $f(V_r) = 0$, which lies on the semiaxis $(V_+, +\infty)$.

Proof. A calculation of the derivative $f'(V)$ and elimination of the parameter α using the equation $f(V) = 0$ give

$$f'(V)|_{f(V)=0} = -\frac{(V+a/2)(V-V_-)(V-V_+)}{V(V+a)} \tag{5.3}$$

Note that $V_- < -a$, i.e. this root does not lie in the permissible range of variation of V. Moreover, $V_+ > 0$ for any values of a. The sign of the derivative $f'(V)$ at the points $f(V) = 0$ is determined from relation (5.3) and is as follows:

$$f'(V)|_{f(V)=0} > 0 \quad \text{for} \quad V \in (-a, -a/2) \cup (0, V_+)$$
$$f'(V)|_{f(V)=0} < 0 \quad \text{for} \quad V \in (-a/2, 0) \cup (V_+, +\infty)$$

Note that $f(-a) = f(0) = -1 < 0$ and $f(V) \to -\infty$ as $V \to +\infty$. Hence, by choosing the left limit V_l from the intervals $(-a, -a/2)$ or $(0, V_+)$ and determining the constant from the equation $f(V_l) = 0$, we can guarantee the existence of the point Vr in the ranges $(-a/2, 0)$ or V_+, $+\infty$, respectively. But in the first case the section $[V_l, V_r]$ necessarily contains the point $V = -a/2$, and of course, does not satisfy the above-mentioned requirements. The second possibility remains, which is also indicated in the formulation of the lemma.

The process of constructing a periodic solution of Eq. (5.2) reduces to the following. For a specified $a \neq 0$ we choose an arbitrary value of V_l from the interval stipulated in Lemma 2. From this, by virtue of Lemma 2, we determine the values of the constants α and V_r. The function $\varphi(V)$ in the interval $[V_l, V_r]$ is monotonic and is calculated by the quadrature

$$\varphi(V)=\int_{V_l}^{V}\frac{(V+a/2)dV}{2(V+a)\sqrt{f(V)}}$$

The inverse function $V(\varphi)$ is extended to all values of φ as even periodic with half-period $T = \varphi(V_r)$.

In order for the solution obtained to be continuous over the whole (r, θ) plane we must require that the function $V(\varphi)$ must be periodic in φ with period 2π, which is equivalent to the equality

$$NT=\pi \tag{5.4}$$

with a certain natural number N. Equation (5.4) was checked numerically. It turned out that by choosing the parameters a and V_l it can be satisfied with $N = 1, 2, 3, \ldots$.

The gas flow pattern in the solution obtained is illustrated in Fig. 2. For $a = 4$ the left limit V_l can be chosen arbitrarily from the interval (0; 0.236). The choice of $V_l = 0.110$ ensures continuity in the plane of the solution with $N = 8$. The invariant streamline is shown in Fig. 2(a). It can be seen that it is invariant under rotation by an angle that is a multiple of $\pi/4$. The contact characteristics of the two families (4.3) and (4.5) are constructed in accordance with the previously mentioned algorithm. Their intersection gives the world line of a particle, the projection of which onto the (r, θ) plane (the trajectory) is given in Fig. 2(b).

The case $\beta = 0$. As shown above, in this case integral (4.8) degenerates into a pair or straight lines $Z = 0$ and $Z = aV$. The first of these gives a vacuum state and is of no interest. Substitution of the second into the remaining equation of (3.6) gives an equation whose solution is expressed explicitly. Hence, we obtain an explicit solution of system (3.6) which, by virtue of (4.6), describes the gas motion with a constant invariant vorticity

$$U=\chi\sin 2\varphi,\qquad V=\alpha+\chi\cos 2\varphi$$
$$Z=a(\alpha+\chi\cos 2\varphi),\quad \chi=\sqrt{\alpha^2-1+a^2} \tag{5.5}$$

A solution exists when $a^2 + \alpha^2 \geqslant 1$. This tracks the difference in the nature of the flow quite well when $a > 1$, $a < 1$ and $a = 1$, which was mentioned earlier. In fact, when $a < 1$ for any φ we obtain $Z > 0$, and of course, the solution (5.5) is "physical" over the whole plane. When $a = 1$ we have the separate value $\varphi_{\rm vac} = \pi/2 + \pi k$ (k is an integer) for which $Z = 0$. Hence, there is a vacuum line $\varphi = \varphi_{\rm vac}$ in the flow. In the initial variables the straight line $x + ty = 0$ corresponds to the vacuum line. Finally, when $a > 1$ the function z is only positive in the sectors

$$S_+=\{-\varphi_{\rm vac}<\varphi<\varphi_{\rm vac},\quad \pi-\varphi_{\rm vac}<\varphi<\pi+\varphi_{\rm vac}\}$$
$$\varphi_{\rm vac}=\mathrm{arctg}((\alpha+\chi)/\sqrt{a^2-1})$$

At the boundary of these sectors it vanishes and is negative outside them. Extending the function Z continuously to zero outside S_+, we obtain a solution, continuous over the whole plane, describing the

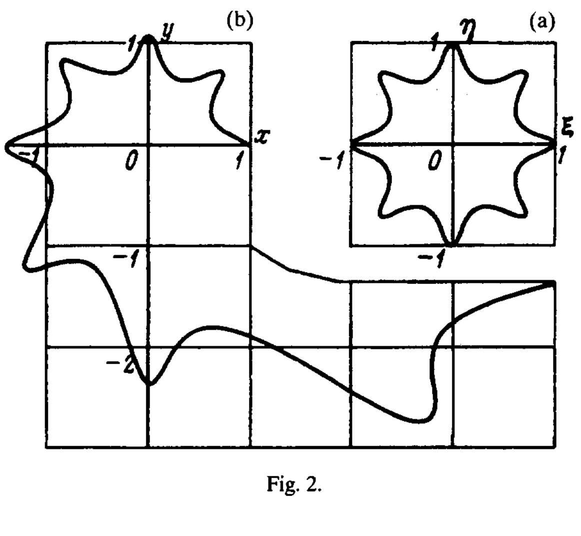

Fig. 2.

gas flow with vacuum regions. The change to the initial variables is made by making the replacement $\theta = \varphi + a \operatorname{arctg} t$.

In Cartesian coordinates $\xi = \lambda \cos\varphi$, $\eta = \lambda \sin\varphi$ the equation of the invariant streamline (4.1) in solution (5.5) has the form

$$(\alpha+\chi)\xi^2 + (\alpha-\chi)\eta^2 = 1/\sqrt{a} \tag{5.6}$$

When $a < 1$ streamline (5.6) is an ellipse, when $a = 1$ it is pair of straight lines and when $a > 1$ it is a hyperbola. When $a > 1$ the asymptotes of hyperbola (5.6) coincide with the boundaries of the vacuum zone.

The hyperbolicity condition (2.5) on solutions of the form (3.1) can be rewritten as

$$Z < Q^2 = U^2 + V^2 \tag{5.7}$$

Flow of the form (3.1), for which inequality (5.7) is satisfied over the whole (λ, φ) plane will henceforth be conditionally called "supersonic". Correspondingly, if the inverse inequality $Z > Q^2$ is satisfied over the whole (λ, φ) plane, this flow will be called "subsonic". Finally, if the inequality $Z < Q^2$ is satisfied on one part of the plane while on the other $Z > Q^2$, this flow will be called "transonic". Here the transition line from "subsonic" to "supersonic" flow is the "sonic line".

A check of inequalities (5.7) for solution (5.5) shows that in the plane of the parameters (a, α) the region of "transonic" flows is defined by the inequality $(1 - a^2) \times (1 - 2a\alpha) < 0$ (shown hatched in Fig. 3). In this case the "sonic line" will be the line

$$\varphi = \varphi_c, \quad \cos 2\varphi_c = (2\alpha^2 - a\alpha + a^2 - 1)\chi^{-1}(a - 2\alpha)^{-1}$$

At the boundary of the "transonic" region we have the following: gas flows that are "supersonic" over the whole (λ, φ) plane correspond to the curves AD, BE and BF (Fig. 3), with the exception of the straight lines $\varphi_c = \pi/2$ for AD and BE and $\varphi_c = 0$ for BF, on which the velocity of sound is reached. "Subsonic" gas flow over the whole plane corresponds to curves AB and BC, with the exception of the straight lines $\varphi_c = 0$ for AB and $\varphi_c = \pi/2$ for BC, which are also "sonic lines". For cases when the parameters (a, α) are chosen from BC or BE, the "sonic line" coincides with the vacuum line, which occurs when $a = 1$. "Sonic" flow over the whole (λ, φ) plane corresponds to points A, B and C, when $Z = Q^2$. Finally, "supersonic" flow corresponds to the regions indicated by numbers 1 and 2 in Fig. 3, while region 3 corresponds to "subsonic" flow.

Note that in solution (5.5) the invariant pressures p and density ρ are constant along the invariant streamline (5.6). Reverting to the "physical" variables we obtain that in solution (5.5) the pressure p and the density ρ along the contact characteristic (4.3) depend only on the time.

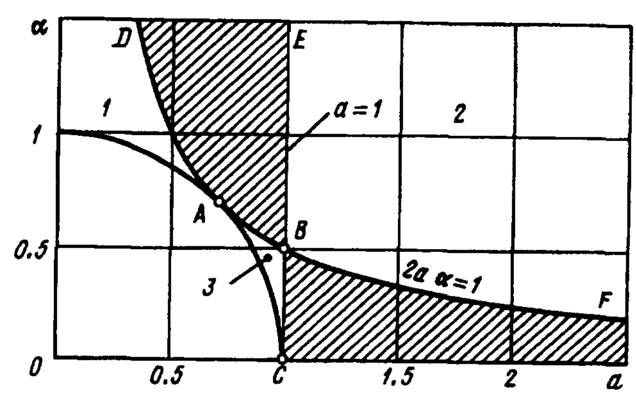

Fig. 3.

6. THE SIMPLE SOLUTION

Below we will consider one of the possible invariant solutions of rank 0 for system (1.3). It is constructed in the two-dimensional $\{H_1, H_2 + kH_3\}$ subalgebra and of course, is identical with the $\{\partial_\varphi\}$-invariant solution of Eqs (3.2). In addition this solution can also be obtained directly from the gas=dynamic equations (1.1) as a "single-step" solution. Hence, we are dealing with a submodel which, in view of the Lie–Ovsyannikov–Talyshev lemma can simultaneously be regarded as "single-step", "two-step" and "three-step".

To obtain this solution we must assume the functions U, V, P, R and s from (3.1) to be independent of φ. Substituting into (3.1) we obtain the solution

$$u=\frac{tr}{t^2+1},\quad v=\frac{Vr}{t^2+1},\quad p=\frac{Pr^k}{(t^2+1)^{2+k/2}},\quad \rho=\frac{Rr^{k-2}}{(t^2+1)^{k/2}}$$

$$S=\frac{S_0r^{4-k}}{(t^2+1)^{2-k/2}},\quad c^2=\frac{Zr^2}{(t^2+1)^2} \tag{6.1}$$

in which the constants V, P, R, S_0 and Z are related as follows:

$$V^2=1+kP/R,\qquad Z=2P/R,\qquad P=S_0R^2$$

By virtue of relation (4.6) the invariant vorticity Ω in this solution is constant.

The trajectory of a particle which starts at $t = 0$ from the position (r_0, θ_0) is given by the formulae

$$r=r_0\sqrt{t^2+1},\quad \theta=\theta_0+V\operatorname{arctg}t$$

Hence, the trajectory of an arbitrary particle is obtained from a single standard particle with $r_0 = 1$, $\theta_0 = 0$ by a homothetic transformation with coefficient r_0 and anticlockwise rotation by an angle θ_0.

To find the sonic characteristics $C^\pm$ of Eqs (1.1) in the form $h(t, r, \theta) = \text{const}$ we must solve the equation

$$h_t+uh_r+r^{-1}h_\theta\pm c\sqrt{h_r^2+r^{-2}h_\theta^2}=0 \tag{6.2}$$

The characteristic conoid is the geometrical location of the characteristics of Eqs (6.2) (the bicharacteristics of the gas-dynamic equations), emerging from the point $(t, r, \theta) = (0, r_0, \theta_0)$. In the solution (6.1) its equation is

$$\left(\ln\frac{r}{r_0\sqrt{t^2+1}}\right)^2+(\theta-\theta_0-V\operatorname{arctg}t)^2=Z\operatorname{arctg}^2t$$

The equations of the sonic characteristics $C^\pm$ in solution (6.1), passing through the curve $r_0 = f(\theta_0)$ at $t = 0$, are given parametrically by the formulae

$$r=f(\theta_0)\sqrt{t^2+1}e^{k_1\operatorname{arctg}t},\quad \theta=\theta_0+(V-k_2)\operatorname{arctg}t$$

$$k_1=\pm\sqrt{Z}f/\sqrt{f^2+f'^2},\quad k_2=\pm\sqrt{Z}f'/\sqrt{f^2+f'^2}$$

A feature of the solution is the existence of a "dead zone" in the neighbourhood of the origin of coordinates, into which no sonic perturbations penetrate at any time. The size of the "dead zone" depends on the initial position of the source of sonic perturbations.

I wish to thank L. V. Ovsyannikov for his comments and interest.

This research was supported financially by the Russian Foundation for Basic Research (99-01-00523) and the Council of Leading Scientific Schools (95-15-96283).

REFERENCES

1. OVSYANNIKOV, L. V., The PODMODELI program. Gas dynamics. *Prikl. Mat. Mekh.*, 1994, **58**, 4, 30–55.
2. KHABIROV, S. V., A non-stationary invariant solution of the equations of gas dynamics, describing the expansion of a gas to a vacuum. *Prikl. Mat. Mekh.*, 1998, **52**, 6, 967–975.
3. OVSYANNIKOV, L. V., Groups and invariant-group solutions of differential equations. *Dokl. Akad. Nauk SSSR*, 1958, **118**, 3, 439–442.
4. OVSYANNIKOV, L. V., *Group Analysis of Differential Equations*. Nauka, Moscow, 1978.
5. NIKOL'SKII, A. A., Invariant transformations of the equations of motion of an ideal gas for special cases. *Inzh. Zh.*, 1963. **3**, 1, 140–142.
6. OVSYANNIKOV, L. V., The hierarchy of invariant submodels of differential equations. *Dokl. Ross. Akad. Nauk*, 1998, **361**, 6, 740–742.
7. OVSYANNIKOV, L. V., Plane gas flows with closed streamlines. *Dokl. Ross. Akad. Nauk*, 1998, **361**, 1, 51–53.
8. CHERNYI, G. G., Plane steady self-similar vortex flows of an ideal fluid. *Izv. Ross. Akad. Nauk, MZhG*, 1997, 4, 39–53.
9. OVSYANNIKOV, L. V., *Lectures on the Fundamentals of Gas Dynamics*. Nauka, Moscow, 1981.

Translated by R.C.G.